

NOTE ON AN EXPANSION OF THE PRODUCT OF TWO OBLONG ARRAYS.

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(1) The long-known expansion of the product of two oblong arrays takes the form of a sum of products of pairs of determinants. The expansion here brought forward takes on the other hand the form of an aggregate of *single* determinants. Of these last the first is itself a product-determinant, and through being bordered gives rise to all the others. Taking the case of two 3-by-5 arrays we have as an example of it

$$\begin{vmatrix} a_1 & a_2 & \dots & a_5 \\ b_1 & b_2 & \dots & b_5 \\ c_1 & c_2 & \dots & c_5 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & \dots & a_5 \\ \beta_1 & \beta_2 & \dots & \beta_5 \\ \gamma_1 & \gamma_2 & \dots & \gamma_5 \end{vmatrix}$$

$$= \begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix} - \begin{vmatrix} P_1 & P_2 & P_3 & a_4 \\ Q_1 & Q_2 & Q_3 & b_4 \\ R_1 & R_2 & R_3 & c_4 \\ a_4 & \beta_4 & \gamma_4 & . \end{vmatrix} - \begin{vmatrix} P_1 & P_2 & P_3 & a_5 \\ Q_1 & Q_2 & Q_3 & b_5 \\ R_1 & R_2 & R_3 & c_5 \\ a_5 & \gamma_5 & \beta_5 & . \end{vmatrix} + \begin{vmatrix} P_1 & P_2 & P_3 & a_4 & a_5 \\ Q_1 & Q_2 & Q_3 & b_4 & b_5 \\ R_1 & R_2 & R_3 & c_4 & c_5 \\ a_4 & \beta_4 & \gamma_4 & . & . \\ a_5 & \beta_5 & \gamma_5 & . & . \end{vmatrix}$$

where $[P_1 Q_2 R_3]$ is the product of $|a_1 b_2 c_3|$ and $|a_1 \beta_2 \gamma_3|$. Had the number of given columns being *six* instead of five, there would have been four additional terms in the expansion, all of them involving the elements with the suffix 6, one like those already obtained of the 4th order, two like those of the 5th order, and one of the 6th order namely,

$$\begin{vmatrix} P_1 & P_2 & P_3 & a_4 & a_5 & a_6 \\ Q_1 & Q_2 & Q_3 & b_4 & b_5 & b_6 \\ R_1 & R_2 & R_3 & c_4 & c_5 & c_6 \\ a_4 & a_5 & a_6 & . & . & . \\ \beta_4 & \beta_5 & \beta_6 & . & . & . \\ \gamma_4 & \gamma_5 & \gamma_6 & . & . & . \end{vmatrix} .$$

The proof is quite simple. We have only to note in the first place that the product of the two arrays is expressible as a single determinant

$$\begin{vmatrix} P_1 & P_2 & P_3 & a_4 & a_5 \\ Q_1 & Q_2 & Q_3 & b_4 & b_5 \\ R_1 & R_2 & R_3 & c_4 & c_5 \\ a_4 & \beta_4 & \gamma_4 & -1 & . \\ a_5 & \beta_5 & \gamma_5 & . & -1 \end{vmatrix},$$

the latter being transformable into the former by adding to each of the first three rows a_4 times the 4th row and a_5 times the 5th row. Then we partition this determinant into four, namely, one representing all the terms of it containing both of the elements in its (4,4)th and (5,5)th places, one representing all the terms containing one of these elements without the other, and one representing all the terms containing neither.

(2) The general theorem may be formulated thus:

The product of two m-by-n arrays A, B, is expressible as an aggregate of single determinants, the first of which is the product of the first k columns of the arrays, and the others are formed from this by bordering, namely, bordering first in every way with one of the remaining columns from A and the corresponding column from B, secondly, with two of the remaining columns from A and the corresponding two from B, and so on, those having an odd number of lines in the border being negative and the others positive.

The number of terms in the expansion is evidently

$$(n-k)0 + (n-k)_1 + \dots + (n-k)_{n-k} \\ \text{i. e. } 2^{n-k}$$

(3) The relation between the old expansion and the new is not at all complicated, each term in the latter being the equivalent of a group of terms in the former. Thus, in the example with which we started, where $n, m, k = 5, 3, 3$, the apportionment of the 10 product-terms of the old expansion among the four terms of the new expansion is

$$1 + 4 + 4 + 1;$$

and when $n, m, k = 6, 3, 3$ the apportionment of the 20 among the 8 (that is, 6_3 among 2^{6-3}) is

$$1 + 3 + 3 + 3 + 3 + 3 + 3 + 1.$$

If, further, we group the latter terms according to the number of lines in a border, this takes the form

$$1 + 3 \cdot 3 + 3 \cdot 3 + 1,$$

and we have the verification provided by the known theorem regarding

the expression of a combinatorial as a sum of products of pairs of combinatorials, namely,

$$(n)_m = (k)_m + (n-k)_1(k)_{m-1} + (n-k)_2(k)_{m-2} + \dots$$

(4) The single determinant used at the end of §1 as the equivalent of a product of two oblong arrays is historically interesting. Its counterpart, the similar expression for two *square* arrays, was first used by Spottiswoode in 1853 in bringing forward* Sylvester's theorem of the year before in regard to the multiplicity of form of the product of two determinants; and as we have pointed out elsewhere† an extreme case of it must have been used by Sylvester himself, namely, the case

$$\begin{vmatrix} a_1 & a_2 & \dots & a_5 \\ b_1 & b_2 & \dots & b_5 \\ c_1 & c_2 & \dots & c_5 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_5 \\ \beta_1 & \beta_2 & \dots & \beta_5 \\ \gamma_1 & \gamma_2 & \dots & \gamma_5 \end{vmatrix} = \begin{vmatrix} . & . & . & a_1 & a_2 & \dots & a_5 \\ . & . & . & b_1 & b_2 & \dots & b_5 \\ . & . & . & c_1 & c_2 & \dots & c_5 \\ a_1 & \beta_1 & \gamma_1 & -1 & . & \dots & . \\ a_2 & \beta_2 & \gamma_2 & . & -1 & \dots & . \\ . & . & . & . & . & . & . \\ a_5 & \beta_5 & \gamma_5 & . & . & \dots & -1 \end{vmatrix}.$$

With this before us it is important to note that this expression for the product of two oblong arrays and the Binet-Cauchy expression of 1812 are the two extremes of a series of such expressions, and that the one used by us at the end of §1 is an example of the *third* of the series.‡

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* *Crelle's Journ.*, li, pp., 238-248.

† *Hist. of Dets.*, ii, pp., 199-200.

‡ Other related papers are:

Muir, T., "On a determinant formed by bordering the product of two determinants," *Messenger of Math.*, xi (1882), pp. 161-165.

Nanson, E. J., "On partial compounds," *Messenger of Math.*, xxvii (1898), pp. 17-19.